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Starlikeness with respect to a boundary point and Julia's theorem in $\mathbb{C}^n$ ☆

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ABSTRACT

In the paper necessary and sufficient conditions for the boundary starlikeness of holomorphic mappings in $\mathbb{C}^n$ are given. In the proof an n-dimensional version of Julia's theorem is used.

Let $\Omega \subset \mathbb{C}^n$, $0 \in \overline{\Omega}$, be a set such that the segment $(0, w]$ is included in Ω for every $w \in \Omega$. Then Ω is called starlike with respect to its point $w = 0$ (shortly starlike), if $0 \in \Omega$ and starlike with respect to its boundary point $w = 0$ (shortly boundary starlike), if $0 \in \partial\Omega$.

Let $\mathbb{B}^n$ denote the open unit ball $\{z \in \mathbb{C}^n\colon \|z\|^2 = \langle z, z\rangle < 1\}$, where $\langle z, w\rangle$ means the inner product of vectors $z, w \in \mathbb{C}^n$.

Let us denote by $\mathcal{ST}$ the family of normalized starlike mappings that is all mappings f, $f(0) = 0$, $Df(0) = I$, that f maps biholomorphically $\mathbb{B}^n$ onto the starlike domain $f(\mathbb{B}^n)$. This family was considered by many authors (see, for instance, the monographs [2,3,5]). Below we remind the main starlikeness theorem, obtained independently by K. Kikuchi [4], T. Matsuno [8] and T.J. Suffridge [11] (see also [12,13]).

Theorem K-M-S. *A locally biholomorphic mapping $f : \mathbb{B}^n \to \mathbb{C}^n$, $f(0) = 0$, $Df(0) = I$, belongs to $\mathcal{ST}$, if and only if*

$$\operatorname{Re}\langle (Df(z))^{-1} f(z), z\rangle > 0, \quad z \in \mathbb{B}^n \setminus \{0\}.$$

Below we define a family of normalized boundary starlike mappings.

Definition. Let us denote by $\mathcal{S}^\partial$ the family of all biholomorphic mappings $f : \mathbb{B}^n \to \mathbb{C}^n$ with the following properties:

(i) $f(\mathbb{B}^n)$ is a starlike domain with respect to boundary point $w = 0$,
(ii) $f(0) = \mathbf{e} = (1, 0, \dots, 0)$,
(iii) there exists the limit $\lim_{\mathbb{B}^n \ni z \to \mathbf{e}} f(z) = 0$ and for every neighbourhood $U_\mathbf{e}$ of the point $\mathbf{e}$ there exists a number $\delta > 0$, such that $f(U_\mathbf{e} \cap \mathbb{B}^n)$ includes the open segment $(0, \delta\mathbf{e})$.

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This definition of boundary starlike mappings is similar to a definition given by M.S. Robertson [9] in one-dimensional case.

Example. Let

$$f(z) = \frac{\mathbf{e} - z}{1 + z^1}, \quad z = (z^1, z^2, \dots, z^n) \in \mathbb{B}^n.$$

Then f biholomorphically maps $\mathbb{B}^n$ onto the domain

$$\Omega = \{ w = (w^1, w^2, \dots, w^n) \in \mathbb{C}^n \colon \Re w^1 > \| (w^2, \dots, w^n) \|^2 \}$$

and the conditions (i)–(iii) are fulfilled. Thus $f \in \mathcal{S}^\partial$.

A natural question is the existence of a condition for boundary starlikeness of locally biholomorphic mappings, similar to the K-M-S result. Our solution of this problem is based on the Julia's theorem in $\mathbb{C}^n$.

Let $\mathbb{E}_k$, $k > 0$, be the set of all $z = (z^1, z^2, \dots, z^n) \in \mathbb{B}^n$ satisfying the inequality:

$$|1 - z^1|^2 < k(1 - \|z\|^2).$$

This inequality is the same as

$$\frac{|z^1 - (1-r)|^2}{r^2} + \frac{\|(z^2, \dots, z^n)\|^2}{r} < 1,$$

where $r = k(1+k)^{-1} \in (0,1)$. Thus $\mathbb{E}_k$ is an ellipsoid in $\mathbb{B}^n$ that has $\mathbf{e}$ as a boundary point, has its center at $(1-r)\mathbf{e}$, has radius r in $\mathbf{e}$-direction, and radii $\sqrt{r}$ in the directions orthogonal to $\mathbf{e}$.

Julia's Theorem. *(See [10].) Let $q = (q^1, \dots, q^n)$ map holomorphically $\mathbb{B}^n$ into itself. If*

$$L = \liminf_{z \to \mathbf{e}} \frac{1 - \|q(z)\|^2}{1 - \|z\|^2} < \infty,$$

then

$$\frac{|1 - q^1(z)|^2}{1 - \|q(z)\|^2} \leqslant L \frac{|1 - z^1|^2}{1 - \|z\|^2}, \quad z \in \mathbb{B}^n.$$

Let us observe that choosing $z = 0$ in the above inequality, we obtain that $L > 0$. Thus we have also the inclusion $q(\mathbb{E}_k) \subset \mathbb{E}_{kL}$, $k > 0$.

Let us recall that the proof of the K-M-S theorem started with the property that starlikeness of $f(\mathbb{B}^n)$ is preserved on $f(r\mathbb{B}^n)$, $0 < r < 1$. We will show that there holds a similar property for boundary starlikeness, if we replace balls $r\mathbb{B}^n$, $0 < r < 1$, by ellipsoids $\mathbb{E}_k$, $k > 0$.

Let us denote $\Omega = f(\mathbb{B}^n)$ and $\Omega_k = f(\mathbb{E}_k)$ for $k > 0$.

Theorem 1. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a biholomorphic mapping with the above properties* (ii)–(iii). *Then $f \in \mathcal{S}^\partial$, if and only if every domain Ω_k, $k > 0$, is starlike w.r.t. the boundary point zero.*

Proof. Assume that $f \in \mathcal{S}^\partial$ and fix arbitrarily $k > 0$. Then, it follows from (iii) that $0 \in \partial\Omega_k$. Hence it is sufficient to prove that for every $t \in (0,1)$ there holds the inclusion

$$t\Omega_k \subset \Omega_k.$$

To show the above inclusion let us define the mapping

$$w(t,z) = f^{-1}(tf(z)), \quad z \in \mathbb{B}^n, \ t \in (0,1]$$

and observe that $w(t,\cdot) : \mathbb{B}^n \to \mathbb{B}^n$ and is holomorphic if $t \in (0,1]$.

From now on let $t \in (0,1)$ be arbitrarily fixed. In this case we define the sequences

$$w_m = t^m \mathbf{e}, \qquad z_m = f^{-1}(w_m), \qquad c_m = \frac{1 - \|w(t, z_m)\|}{1 - \|z_m\|}, \quad m \in \mathbb{N}.$$

Then

$$z_{m+1} = f^{-1}(w_{m+1}) = f^{-1}(tw_m) = f^{-1}\big(tf(z_m)\big) = w(t, z_m),$$
$$c_m = \frac{1 - \|z_{m+1}\|}{1 - \|z_m\|}, \quad m \in \mathbb{N}.$$

Moreover from (iii), we have also that $z_m \to \mathbf{e}$, if $m \to \infty$. Hence

$$\lim_{m\to\infty} c_1 \cdot \cdots \cdot c_m = \lim_{m\to\infty} \frac{1 - \|z_{m+1}\|}{1 - \|z_1\|} = 0,$$

because $\|z_m\| \to 1$, if $m \to \infty$. Therefore, the sequence (c_m) has a convergent subsequence (c_{m_ν}) with a limit $\lambda = \lambda_t \in [0, 1]$, and consequently,

$$\lim_{\nu\to\infty} \frac{1 - \|w(t, z_{m_\nu})\|}{1 - \|z_{m_\nu}\|} = \lambda \in [0, 1].$$

Hence and by the equality $\lim_{m\to\infty} z_m = \mathbf{e}$, we obtain that

$$\liminf_{z\to\mathbf{e}} \frac{1 - \|w(t, z)\|^2}{1 - \|z\|^2} \leqslant \lambda.$$

Thus, in view of Julia's theorem, we have

$$\frac{|1 - w^1(t, z)|^2}{1 - \|w(t, z)\|^2} \leqslant \lambda \frac{|1 - z^1|^2}{1 - \|z\|^2}, \quad w(t, z) = \big(w^1(t, z), \ldots, w^n(t, z)\big),\ z \in \mathbb{B}^n,$$

hence $\lambda \in (0, 1]$ and $w(t, \cdot)(\mathbb{E}_k) \subset \mathbb{E}_{k\lambda}$ for $k > 0$. Consequently, we have the inclusion $w(t, \cdot)(\mathbb{E}_k) \subset \mathbb{E}_k$ for $k > 0$. This and t arbitrary in $(0, 1)$ implies the inclusion $t\Omega_k \subset \Omega_k$ for every $t \in (0, 1)$.

Now let us assume that every domain Ω_k, $k > 0$, is starlike w.r.t. the boundary point $0 \in \partial\Omega_k$. We will show that Ω is a starlike domain w.r.t. boundary point $0 \in \partial\Omega$. To do that, let us choose arbitrarily a fixed point $w' \in \Omega$. Then the point $z' = f^{-1}(w')$ belongs to an ellipsoid $\mathbb{E}_{k'}$, $k' > 0$, because $\bigcup_{k>0} \mathbb{E}_k = \mathbb{B}^n$. Hence $w' \in \Omega_{k'}$, which implies in view of the assumption that the segment $(0, w']$ is included in $\Omega_{k'}$, so

$$\big(0, w'\big] \subset \Omega = \bigcup_{k>0} \Omega_k.$$

Thus Ω is a starlike domain w.r.t. the point $0 \in \partial\Omega$, hence $f \in \mathcal{S}^\partial$. □

Remark 1. In Theorem 1 we can replace the starlikeness w.r.t. the boundary point zero of every domain Ω_k, $k > 0$, by the starlikeness of the closure $\overline{\Omega_k}$, w.r.t. its point zero.

Proof. Let $k > 0$ be arbitrarily fixed and Ω_k be a starlike domain w.r.t. the boundary point $0 \in \partial\Omega_k$. If the closure $\overline{\Omega_k}$ was not starlike w.r.t. its point $0 \in \overline{\Omega_k}$, then there would exist a point $w' \in \partial\Omega_k$ such that the segment $(0, w']$ was not included in $\overline{\Omega_k}$. Thus $w = sw' \notin \overline{\Omega_k}$ for an $s \in (0, 1)$ and consequently, w has a neighbourhood V_w such that $V_w \cap \overline{\Omega_k} = \emptyset$. On the other hand, there exists a sequence of points $w_m \in \Omega_k$, convergent to w'. Clearly, $sw_m \to w$, if $m \to \infty$, so $sw_m \in V_w$ for sufficiently large m and consequently, $sw_m \notin \overline{\Omega_k}$ for such m. This gives that the segment $(0, w_m]$ is not included in Ω_k although $w_m \in \Omega_k$ for sufficiently large m, which contradicts our assumption.

Now let us assume that the closure $\overline{\Omega_k}$ is starlike w.r.t. its point $0 \in \overline{\Omega_k}$. If the domain Ω_k was not starlike w.r.t. the boundary point $0 \in \partial\Omega_k$, then there would exist a point $w'' \in \Omega_k$ such that the segment $(0, w'')$ was not included in Ω_k. Let w be a point from $(0, w'') \cap \partial\Omega_k$. On the other hand, there exists a sequence of points $w_m \notin \overline{\Omega_k}$ convergent to w. Consider a neighbourhood $V_{w''} \subset \Omega_k$ and rays $I_m = \{tw_m\colon\ t > 0\}$. The rays I_m have nonempty intersections with $V_{w''}$ for sufficiently large m, hence there is a point $\widetilde{w} \in V_{w''}$ such that the segment $(0, \widetilde{w})$ is not included in $\overline{\Omega_k}$. Thus $\overline{\Omega_k}$ is not starlike w.r.t. the point $0 \in \overline{\Omega_k}$, which contradicts our assumption. □

Next theorem will be preceded by a lemma including a characterization of domains $\Omega_k = f(\mathbb{E}_k)$, $k > 0$, in terms of the outward normals to $\partial\Omega_k \setminus \{0\}$.

Let $D \subset \mathbb{C}^n$ be a domain, $f : D \to \mathbb{C}^n$ be a locally biholomorphic mapping, E be a subdomain of D such that the closure $\overline{E}$ is a smooth manifold with boundary and let $G = f(E)$, $z \in D \cap \partial E$. We will say that the unit vector $n(f(z))$ is a vector of the outward normal to ∂G at the point $f(z)$, if $n(f(z))$ is normal to ∂G and for every sufficiently small neighbourhood U_z of the point z there is $\delta > 0$ such that $f(z) + sn(f(z)) \notin f(E \cap U_z)$ for every $s \in (0, \delta)$. Of course, if f is a biholomorphic mapping, then the outward normal to ∂G is determined uniquely by the point $w = f(z) \in \partial G$ at which this normal is evaluated. Hence, for biholomorphic mapping f, we will use both notions: $n(f(z))$ and $n(w)$ for $w = f(z) \in \partial G$ equivalently.

Lemma. *Let $f:\mathbb{B}^n\to\mathbb{C}^n$ be a biholomorphic mapping with the above properties* (ii)–(iii) *and let $k>0$, $w^0\in\partial\Omega_k\setminus\{0\}$ be arbitrarily fixed. Then, for every $q\in(0,1)$ there exists an $\varepsilon>0$ such that among the bounded cones*

$$K^+(w^0,q,\varepsilon)=\{w\colon \Re\langle w-w^0,n(w^0)\rangle>q\|w-w^0\|,\ \|w-w^0\|<\varepsilon\},$$
$$K^-(w^0,q,\varepsilon)=\{w\colon \Re\langle w-w^0,-n(w^0)\rangle>q\|w-w^0\|,\ \|w-w^0\|<\varepsilon\}$$

the first is disjoint with the domain Ω_k and the latter is included in Ω_k.

Proof. Let $k>0$ be arbitrarily fixed. We can easily check that $\partial\mathbb{E}_k\setminus\{\mathbf{e}\}$ is a $(2n-1)$-dimensional manifold of C^1 class in $\mathbb{R}^{2n}$. Hence, in view of the fact that f is a C^1-diffeomorphism, $\partial\Omega_k\setminus\{0\}$ is also a $(2n-1)$-dimensional manifold of C^1 class in $\mathbb{R}^{2n}$. Moreover, from theorem of Lusternik (see e.g. [1, Chapter 10, §2.3, Theorem 4]) it follows that for points $w\in\partial\Omega_k\setminus\{0\}$ sufficiently close to w^0 there holds the equality

$$w-w^0=w^\perp-w^0+o(\|w-w^0\|), \tag{1}$$

where $w^\perp$ is the projection of w onto the hyperplane $\mathcal{P}_{w^0}$, tangent to hypersurface $\partial\Omega_k\setminus\{0\}$ at w^0.

Suppose that the former thesis is not true. Then, there exists $q\in(0,1)$ such that for every $\varepsilon>0$, there holds the relation

$$\Omega_k\cap K^+(w^0,q,\varepsilon)\neq\emptyset.$$

Hence, there exists a sequence of points $w_m\in\Omega_k$, convergent to w^0 such that

$$\Re\langle w_m-w^0,n(w^0)\rangle>q\|w_m-w^0\|. \tag{2}$$

Defining for $m\in\mathbb{N}$

$$v_m(t)=w^0+t(w_m-w^0)+(1-t)\|w_m-w^0\|n(w^0),\quad t\in[0,1],$$

we obtain that for every m the function $v_m(t)$ is continuous in $[0,1]$ and $v_m(1)\in\Omega_k$. Since $v_m(0)=w^0+\|w_m-w^0\|n(w^0)\notin\Omega_k$, for sufficiently large m, we get that for every sufficiently large m there exists $t_m\in[0,1)$ such that $v_m(t_m)\in\partial\Omega_k$. Since $v_m=v_m(t_m)\to w^0$ for $m\to\infty$, we have in view of (2) that

$$\Re\langle v_m-w^0,n(w^0)\rangle>(qt_m+1-t_m)\|w_m-w^0\|\geqslant q\|w_m-w^0\|.$$

Hence and by the inequality $\|v_m-w^0\|\leqslant\|w_m-w^0\|$, we get

$$\Re\left\langle\frac{v_m-w^0}{\|v_m-w^0\|},n(w^0)\right\rangle\geqslant q>0. \tag{3}$$

On the other hand, applying (1) to v_m in (3), we obtain

$$\Re\left\langle\frac{v_m-w^0}{\|v_m-w^0\|},n(w^0)\right\rangle=\Re\left\langle\frac{v_m^\perp-w^0}{\|v_m-w^0\|},n(w^0)\right\rangle+\Re\left\langle\frac{o(v_m-w^0)}{\|v_m-w^0\|},n(w^0)\right\rangle.$$

Since the first addend of the above sum equals 0 (vectors $v_m^\perp-w^0$ are orthogonal to $n(w^0)$), the right-hand side tends to 0 if $m\to\infty$. This contradicts inequality (3). Therefore, our supposition was false and the former thesis of lemma is true.

Now let us suppose that the second thesis is not true. Then there exists $q\in(0,1)$ such that for every $\varepsilon>0$ the cone $K^-(w^0,q,\varepsilon)$ is not included in Ω_k. Hence, there exists a sequence of points $w_m\in\mathbb{C}^n\setminus\Omega_k$, convergent to w^0 such that

$$\Re\langle w_m-w^0,(-n(w^0))\rangle>q\|w_m-w^0\|.$$

Defining for $m\in\mathbb{N}$

$$v_m(t)=w^0+t(w_m-w^0)+(1-t)\|w_m-w^0\|(-n(w^0)),\quad t\in[0,1],$$

we obtain that for every m the function $v_m(t)$ is continuous in $[0,1]$ and $v_m(1)\notin\Omega_k$. Since $v_m(0)=w^0+\|w_m-w^0\|(-n(w^0))\in\Omega_k$, for sufficiently large m, we get that for every sufficiently large m there exists $t_m\in(0,1]$ such that $v_m(t_m)\in\partial\Omega_k$. Now, replacing $n(w^0)$ by $-n(w^0)$ in the rest of the proof of the property of the cone $K^+(w^0,q,\varepsilon)$, we obtain a contradiction similarly as above. Thus our supposition was false and consequently for every $q\in(0,1)$ there exists an $\varepsilon>0$ such that $K^-(w^0,q,\varepsilon)\subset\Omega_k$.

This completes the proof of the lemma. □

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Remark 2. Let $k > 0$, $z^0 \in \partial\mathbb{E}_k \setminus \{\mathbf{e}\}$ be arbitrarily fixed. Replacing in the lemma the biholomorphicity of f in $\mathbb{B}^n$ by the local biholomorphicity in $\mathbb{B}^n$ and biholomorphicity in $\mathbb{E}_k$, we obtain, in a similar way, that for every $q \in (0,1)$ there exists an $\varepsilon > 0$ such that the bounded cone

$$K^-\big(f\big(z^0\big), q, \varepsilon\big) = \big\{w\colon \Re\big\langle w - f\big(z^0\big), -n\big(f\big(z^0\big)\big)\big\rangle > q\big\|w - f\big(z^0\big)\big\|, \big\|w - f\big(z^0\big)\big\| < \varepsilon\big\}$$

is included in the domain Ω_k.

Now we shall give an analytic characterization of the boundary starlikeness of biholomorphic mappings.

Theorem 2. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a biholomorphic mapping with the above properties* (ii)–(iii). *Then $f \in \mathcal{S}^\partial$, if and only if*

$$\Re\big\langle f(z), n\big(f(z)\big)\big\rangle \geqslant 0, \quad z \in \mathbb{B}^n. \tag{4}$$

Proof. The proof will be divided into 3 steps.

1. We first prove that the inequality

$$\Re\big\langle f(z), n\big(f(z)\big)\big\rangle > 0, \quad z \in \mathbb{B}^n, \tag{5}$$

implies that $f \in \mathcal{S}^\partial$. To do that it is sufficient to show, in view of Theorem 1, that every domain Ω_k, $k > 0$, is starlike w.r.t. the boundary point 0. Suppose, on the contrary, that there exists a number $k > 0$ such that Ω_k is not starlike w.r.t. boundary point 0. Then there exists a point $w \in \Omega_k$ such that the segment $(0, w)$ is not included in Ω_k. Let us denote $w^0 = wt_0 \in \partial\Omega_k \setminus \{0\}$, where

$$t_0 = \sup\big\{t \in (0,1)\colon tw \notin \Omega_k\big\}.$$

Then $t_0 > 0$, $w^0(1+t) \in \Omega_k$, for a small $t > 0$ and by the lemma, for every $q \in (0,1)$ there exists $\varepsilon > 0$ such that $\Omega_k \cap K^+(w^0, q, \varepsilon) = \emptyset$. Hence, $w^0(1+t) \notin K^+(w^0, q, \varepsilon)$. Consequently $\Re\langle w^0(1+t) - w^0, n(w^0)\rangle \leqslant q\|w^0(1+t) - w^0\|$, for every $q \in (0,1)$. Then q arbitrary in $(0,1)$ implies $\Re\langle w^0, n(w^0)\rangle \leqslant 0$. Since $w^0 = f(z^0)$, where $z^0 \in \partial\mathbb{E}_k \setminus \{\mathbf{e}\} \subset \mathbb{B}^n$, the above inequality coincides with the inequality $\Re\langle f(z^0), n(f(z^0))\rangle \leqslant 0$, which contradicts (5) and consequently, proves that $f \in \mathcal{S}^\partial$.

2. Now we prove that the relation $f \in \mathcal{S}^\partial$ implies inequality (4). Suppose that (4) does not hold at a point $z^0 \in \mathbb{B}^n$. Then there exists an index $k > 0$ such that $z^0 \in \partial\mathbb{E}_k \setminus \{\mathbf{e}\}$, $w^0 = f(z^0) \in \partial\Omega_k \setminus \{0\}$ and $\Re\langle w^0, n(w^0)\rangle < 0$. Since for $t \in (0,1)$

$$\Re\left\langle \frac{tw^0 - w^0}{\|tw^0 - w^0\|}, n\big(w^0\big)\right\rangle = -\Re\left\langle \frac{w^0}{\|w^0\|}, n\big(w^0\big)\right\rangle > 0,$$

so there exists $q \in (0,1)$ such that

$$\Re\left\langle \frac{tw^0 - w^0}{\|tw^0 - w^0\|}, n\big(w^0\big)\right\rangle > q$$

for every $t \in (0,1)$. Consequently, for this $q \in (0,1)$ and every $\varepsilon > 0$ there is $t \in (0,1)$ sufficiently close to 1 such that the point tw^0 belong to the cone $K^+(w^0, q, \varepsilon)$. Therefore, in view of lemma, there exists a neighbourhood V_{tw^0} of the point tw^0 such that $\Omega_k \cap V_{tw^0} = \emptyset$. Hence $tw^0 \notin \overline{\Omega_k}$. However, this is impossible, because in view of Theorem 1 and Remark 1, all sets $\overline{\Omega_k}$, $k \in (0,1)$, are starlike w.r.t. the boundary point 0. Thus, our supposition was false and inequality (4) is true.

3. Finally we prove that inequality (4) implies the relation $f \in \mathcal{S}^\partial$. In view of Theorem 1 and Remark 1 it suffices to prove that every $\overline{\Omega_k}$, $k > 0$, is starlike w.r.t. the point 0. Suppose that an $\overline{\Omega_k}$ is not starlike w.r.t. its boundary point 0. Then there exists $w' \in \overline{\Omega_k}$ (may be $w' \in \partial\Omega_k$) such that the segment $(0, w')$ is not included in $\overline{\Omega_k}$. Hence there is v', $v' \in (0, w')$ but $v' \notin \overline{\Omega_k}$, so there is a neighbourhood $V_{v'}$ disjoint with $\overline{\Omega_k}$. Thus there exists a neighbourhood $V_{w'}$ such that for all $w'' \in V_{w'} \cap \Omega_k$ the intersection of $V_{v'}$ and the segment $(0, w'')$ is nonempty. Moreover, for such w'' there exists a neighbourhood $V_{w''} \subset \Omega_k$ such that for every $w \in V_{w''}$ we have $(0, w) \cap V_{v'} \neq \emptyset$. Choosing a point $w^0 \in (0, w'') \cap \partial\Omega_k$, similarly as in the first step, we have

$$\Re\big\langle w^0, n\big(w^0\big)\big\rangle = 0, \tag{6}$$

in view of (4) and the inequality $\Re\langle w^0, n(w^0)\rangle \leqslant 0$ from the first part of the proof.

Let us denote by Σ the set of all points from $\partial\Omega_k \setminus \{0\}$ satisfying the equality (6). From the above considerations there follows that if $w^0 \in \Sigma$, then there exists (in $\partial\Omega_k \setminus \{0\}$) a neighbourhood Υ_{w^0} of w^0 that (6) is fulfilled for every $w \in \Upsilon_{w^0}$. This gives that Σ is open in $\partial\Omega_k \setminus \{0\}$. Since $\Re\langle w, n(w)\rangle$ is continuous with respect to w, the set Σ is also closed in $\partial\Omega_k \setminus \{0\}$. So Σ is empty or $\Sigma = \partial\Omega_k \setminus \{0\}$.

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We will show that the latter equality is false. By the property (iii) of f, it follows that Ω_k is bounded, hence in view of the smoothness of the surface $\partial\Omega_k \setminus \{0\}$, there exists a ball B with the center at the origin such that $\Omega_k \subset B$ and that the $(2n-1)$-dimensional real sphere $S^{2n-1} = \partial B$ is tangent to $\partial\Omega_k$ at a point $w = f(z) \in \partial\Omega_k \setminus \{0\}$. Thus the outward normal vector $n(w)$ to the hypersurface $\partial\Omega_k$ is the same as the outward normal vector to the sphere S^{2n-1} at the point w, that is $n(w) = w$. Since $\Re\langle w, n(w)\rangle > 0$, we have $\Sigma \neq \partial\Omega_k \setminus \{0\}$. Consequently, $\Sigma = \emptyset$, which contradicts equality (6).

Therefore, our supposition is false and consequently, every $\overline{\Omega_k}$, $k > 0$, is starlike w.r.t. the point 0. This shows that $f \in \mathcal{S}^\partial$ and completes the proof. □

By the proof of Theorem 2, it follows that the equality in (4) never holds. Thus we have the following equivalent theorem:

Theorem 3. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a biholomorphic mapping with the above properties* (ii)–(iii). *Then $f \in \mathcal{S}^\partial$ if and only if the condition* (5) *is fulfilled.*

Theorem 3 yields the following boundary starlikeness analytic condition, corresponding to the condition from K-M-S theorem.

Corollary. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a biholomorphic mapping with the above properties* (ii)–(iii). *Then $f \in \mathcal{S}^\partial$ if and only if for every point $z = (z^1, z^2, \ldots, z^n) \in \mathbb{B}^n$ there holds the following inequality*

$$\Re\left\langle Df(z)^{-1} f(z), \left(\frac{(1-z^1)(z^1 - 1 + \|(z^2, \ldots, z^n)\|^2)}{|1-z^1|^2}, z^2, \ldots, z^n\right)\right\rangle > 0.$$

Proof. Let

$$\varphi(z) = \left|1 - z^1\right|^2 \left(1 - \|z\|^2\right)^{-1} - k, \quad z = \left(z^1, z^2, \ldots, z^n\right) \in \mathbb{B}^n,\ k > 0.$$

Since the hypersurface $\partial\mathbb{E}_k \setminus \{\mathbf{e}\} = \{z \in \mathbb{B}^n\colon\ \varphi(z) = 0\}$ is smooth, we have that $\partial\Omega_k \setminus \{0\} = \{w \in \Omega\colon\ \phi(w) \equiv \varphi(f^{-1}(w)) = 0\}$ is smooth, because the mapping f is biholomorphic. Moreover, the outward normal vector to the boundary $\partial\Omega_k$ at the point $w = f(z)$ has the form

$$n\big(f(z)\big) = \frac{\partial\phi(w)}{\partial w^*} = \left(Df(z)^{-1}\right)^* \left(\frac{\partial\varphi(z)}{\partial z}\right)^*, \quad z \in \mathbb{B}^n,$$

where A^* means the transpose and conjugate of a matrix A (see e.g. [6], compare also [4]). Therefore,

$$\begin{aligned}\big\langle f(z), n\big(f(z)\big)\big\rangle &= \big(n\big(f(z)\big)\big)^* f(z) = \frac{\partial\varphi(z)}{\partial z} Df(z)^{-1} f(z)\\ &= \left\langle Df(z)^{-1} f(z), \left(\frac{\partial\varphi(z)}{\partial z}\right)^*\right\rangle.\end{aligned}$$

Now it is suffice to use Theorem 3 and observe that the column vector $(\frac{\partial\varphi(z)}{\partial z})^*$ has the form

$$\left(\frac{\partial\varphi(z)}{\partial z}\right)^* = \frac{|1-z^1|^2}{(1-\|z\|^2)^2}\left(\frac{(1-z^1)(z^1 - 1 + \|(z^2, \ldots, z^n)\|^2)}{|1-z^1|^2}, z^2, \ldots, z^n\right). \quad \square$$

Remark 3. For $n = 1$, the condition in corollary reduces to the following form

$$\Re\left((1-z)^2 \frac{f'(z)}{f(z)}\right) < 0, \quad z \in \mathbb{B}^1,$$

which is the same as the condition obtained by A. Lecko in [7] in the one-dimensional case.

In the next theorem we will show that the assumption of biholomorphicity of mapping f in Theorem 2 can be replaced in fact by the local biholomorphicity.

Theorem 4. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a locally biholomorphic mapping with the above properties* (ii)–(iii). *If f is injective in the intersection of $\mathbb{B}^n$ and a neighbourhood $U_{\mathbf{e}}$ of $\mathbf{e}$, and if relation* (4) *is fulfilled, then f is biholomorphic in $\mathbb{B}^n$, hence $f \in \mathcal{S}^\partial$.*

Proof. Suppose that f is not injective in $\mathbb{B}^n$. Since f is injective in $\mathbb{B}^n \cap U_{\mathbf{e}}$, the set

$$\{k > 0\colon\ f \text{ is injective in } \mathbb{E}_k\}$$

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is nonempty and its supremum k_0 is finite. Hence f is injective in $\mathbb{E}_{k_0}$, because every two different points from $\mathbb{E}_{k_0}$ belong to an $\mathbb{E}_k$, $k < k_0$, and f is injective in $\mathbb{E}_k$. If k_m tends to k_0 and $k_m > k_0$, then by the definition of the number k_0, we find in every set $\mathbb{E}_{k_m}$ points $z'_m \neq z''_m$ such that $f(z'_m) = f(z''_m)$. Choosing, if necessary, appropriate subsequences, we can assume that $\lim_{m\to\infty} z'_m = z' \in \overline{\mathbb{E}_{k_0}}$ and $\lim_{m\to\infty} z''_m = z'' \in \overline{\mathbb{E}_{k_0}}$.

Now we consider two cases.

1. First let us assume that $z', z'' \in \overline{\mathbb{E}_{k_0}} \setminus \{\mathbf{e}\}$.

Since f is continuous in $\overline{\mathbb{E}_{k_0}} \setminus \{\mathbf{e}\}$, we have $f(z') = f(z'')$. Hence it follows that f is not injective in $\overline{\mathbb{E}_{k_0}} \setminus \{\mathbf{e}\}$. Indeed, in the opposite case we have $z' = z''$, so the equality $f(z'_m) = f(z''_m)$ holds, in particular, for $z'_m \neq z''_m$ sufficiently close to z', which is impossible, because f is injective in a neighbourhood of the point $z' \in \mathbb{B}^n$.

Now we observe that if $f(z^\wedge) = f(z^\vee)$ for a pair of different points $z^\wedge, z^\vee \in \overline{\mathbb{E}_{k_0}} \setminus \{\mathbf{e}\}$, then $z^\wedge, z^\vee \in \partial\mathbb{E}_{k_0} \setminus \{\mathbf{e}\}$. To do it, suppose that $z^\wedge \in \mathbb{E}_{k_0}$ and $z^\vee \in \partial\mathbb{E}_{k_0}$ (the relation $z^\wedge, z^\vee \in \mathbb{E}_{k_0}$ is impossible, because f is injective in $\mathbb{E}_{k_0}$). From our supposition there follows that f maps biholomorphically two disjoint neighbourhoods $U_{z^\wedge}, U_{z^\vee}$, $U_{z^\wedge} \subset \mathbb{E}_{k_0}$ (of the points $z^\wedge$, $z^\vee$ respectively) onto a neighbourhood V_{w^0}, of the point $w^0 = f(z^\wedge) = f(z^\vee)$. Thus, there exist points $\zeta^\wedge \in U_{z^\wedge}$, $\zeta^\vee \in U_{z^\vee} \cap \mathbb{E}_{k_0}$ such that $f(\zeta^\wedge) = f(\zeta^\vee)$ which contradicts the injectivity of f in $\mathbb{E}_{k_0}$.

From Remark 2 it follows that for every $q \in (0,1)$ there exists an $\varepsilon > 0$ such that

$$K^-\big(f(z^\wedge), q, \varepsilon\big), K^-\big(f(z^\vee), q, \varepsilon\big) \subset \Omega_{k_0}.$$

Equivalently,

$$n\big(f(z^\wedge)\big) = -n\big(f(z^\vee)\big),$$

because in the opposite case the cones $K^-(f(z^\wedge), q, \varepsilon)$, $K^-(f(z^\vee), q, \varepsilon)$ have a nonempty intersection for sufficiently small $q \in (0,1)$, in view of the equality $f(z^\wedge) = f(z^\vee)$.

Thus, replacing z in (4) by $z^\wedge$, $z^\vee$ we obtain a contradiction. Consequently, our thesis that f is not injective in $\overline{\mathbb{E}_{k_0}} \setminus \{\mathbf{e}\}$ is false. Hence, the case $z', z'' \in \overline{\mathbb{E}_{k_0}} \setminus \{\mathbf{e}\}$ does not hold.

2. Now let us assume that $z' \in \overline{\mathbb{E}_{k_0}} \setminus \{\mathbf{e}\}$ and $z'' = \mathbf{e}$. Since $w'' = \lim_{m\to\infty} f(z''_m) = 0$, we have $0 = w' = f(z')$, where $z' \in \overline{\mathbb{E}_{k_0}} \setminus \{\mathbf{e}\}$. Obviously, there is $z' \in \partial\mathbb{E}_{k_0} \setminus \{\mathbf{e}\}$. Indeed, if we assume that $z' \in \mathbb{E}_{k_0}$, then we find a neighbourhood $U_{z'} \subset \mathbb{E}_{k_0}$ of the point z' such that $f(U_{z'})$ is a neighbourhood of the point $w = 0$. However, this contradicts the injectivity of f in $\mathbb{E}_{k_0}$ and the fact that $f(z) \to 0$, if $\mathbb{B}^n \ni z \to \mathbf{e}$. Hence, $z' \in \partial\mathbb{E}_{k_0} \setminus \{\mathbf{e}\}$. This gives that there exists a sequence of points $z_m \in \mathbb{E}_{k_0}$, $\lim_{m\to\infty} z_m = z'$ such that $w_m = f(z_m) \to f(z') = 0$, if $m \to \infty$. Of course $z_m \in \mathbb{E}_{k_m}$, $k_m < k_0$, for every m. On the other hand, from the injectivity of f in $\mathbb{E}_{k_m} \subset \mathbb{E}_{k_0}$ and from condition (4) we obtain, similarly as in the proof of Theorem 2, the starlikeness of $\Omega_{k_m} = f(\mathbb{E}_{k_m})$ w.r.t. the boundary point $w = 0$. Thus the segment $(0, w_m)$ is included in Ω_{k_m} for every m. Then, for every m, the curve

$$\Gamma_m = f^{-1}\big((0, w_m)\big) \subset \mathbb{E}_{k_m}$$

connects the point z_m and $\mathbf{e}$ or a point $z^* \in \partial\mathbb{E}_{k_m} \setminus \{\mathbf{e}\}$. Let us observe that the second case is impossible, because $z^* \in \mathbb{E}_{k_0}$ and f is injective in $\mathbb{E}_{k_0}$.

Consider a neighbourhood $U_{z'}$ with radius $2\delta > 0$, $\overline{U_{z'}} \subset \mathbb{B}^n$. Then the length of this part γ_m of the curve Γ_m which joints the point z_m with $\partial U_{z'}$ is greater than δ for sufficiently large m. Moreover, there is a constant $c > 0$, such that the length of the curve $f(\gamma_m)$ is greater than c, for sufficiently large m, because the norm $\|Df(z)\|$ has a positive lower bound for $z \in U_{z'}$. On the other hand, $f(\gamma_m) \subset (0, w_m)$, hence the length of $f(\gamma_m)$ tends to 0, if $m \to \infty$, which yields a contradiction.

This shows that the case $z'' = \mathbf{e}$, $z' \in \overline{\mathbb{E}_{k_0}} \setminus \{\mathbf{e}\}$ also does not hold. Concluding, our results of case 1 and case 2 contradict the relation $z', z'' \in \overline{\mathbb{E}_{k_0}} \setminus \{\mathbf{e}\}$, $z' \neq z''$.

This contradiction shows that our supposition that f is not injective in $\mathbb{B}^n$ is false. This completes the proof. □

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