On Hölder continuity of QC maps

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Goal

Nontechnical survey

Battlefield

Cont.maps $f : G \to G'$**, where** $G, G' \subset \mathbb{R}^n$ $(n \geq 2)$ **are domains.** **(**f **often 1-to-1)**

Notation: $B^n \subset \mathbb{R}^n$ **is the unit ball.**

Liouville

$n \geq 3$ & f **conf.** $\Rightarrow f = g|G$, **where** g **is a Möbius transformation of** $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$

Definition

A homeo f **is K-qc if:**

$\forall$ **(curve family** Γ**)** :

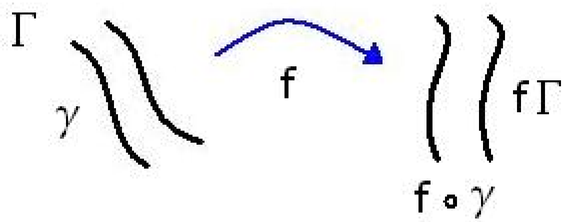
$M(\Gamma)/K \leq M(f\Gamma) \leq KM(\Gamma)$.

Conformal modulus

$M(\Gamma) = \inf_{\rho \in A(\Gamma)} \int_{\mathbb{R}^n} \rho^n \, \mathrm{d}m.$

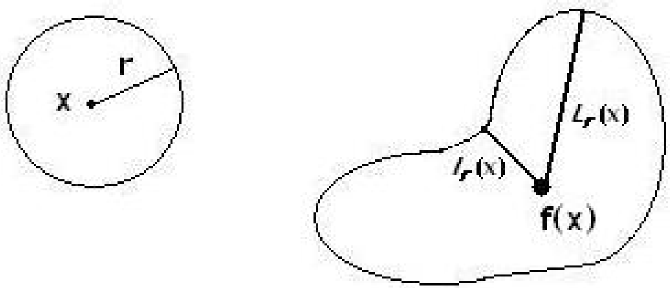
$A(\Gamma) = \{\rho \in \mathbf{Bor}(\mathbb{R}^n) : \rho \geq 0$ &

$\int_\gamma \rho \, \mathrm{d}s \geq 1, \quad \forall$ **locally rectif.** $\gamma \in \Gamma\}$



Geometric meaning: H(x), the linear dilatation:

$\forall K > 1, \exists c(K) (\to 1, when\ K \to 1)$:
f **K-qc**, $x \in G \Rightarrow$
$H(x) = \overline{\lim}_{r \to 0} \frac{L_r(x)}{l_r(x)} \le c(K) < \infty$.



Note

1. $M(\Gamma)$ **is conf. invariant**
2. **1-qc=conf.** $(c(1) = 1)$
3. "$c(K)$" **depends transcendentally on** K

Example

L**-bilip are** $L^{2(n-1)}$**-qc (not conver.).**

$BILIP \subsetneq QC \subsetneq \text{Hölder}\,, K > 1.$

f is Hölder if

$\exists(\alpha \in (0,1], c \geq 1), \forall x, y \in G$:
$|f(x) - f(y)| \leq C|x-y|^{\alpha}$

f is local Hölder if

$\forall(\text{comp. } F \subset G) : f|F$ is Hölder.

Particular case: $\alpha = 1$, **Lipschitz.**

$QC_K(G) = \{f : G \to G = fG : f \, is \, K - qc\}$.

Let $Z_f = \{z \in G : f : G \to G'$
is not diffble at z or $J_f(z) = 0\}$. **Basic fact:**
$m_n(Z_f) = 0$ **if** f **is** K**-qc.**

Two main problems of qc theory

If $Z_f \neq \emptyset$ qcircles may be fractals

Problem 1: Stability theory

In which sense K-qc maps are close to conformal maps when $K \to 1$?

Problem 2: What happens close to Z_f ?

- **local behavior at the points of Z_f**
- **differentiability at the points of Z_f**
- **local int.bility of $|f'(x)|$ close to Z_f**

Theorem (Ahlfors -54)

$f \in QC_K(B^2) \Rightarrow f$ **is Hölder.**

Theorem (Mori -56)

$f \in QC_K(B^2), f(0) = 0, \forall\ x, y \in B^2 \Rightarrow$
$|f(x) - f(y)| \leq 16|x - y|^{1/K}$

Definition $\alpha = K^{1/(1-n)} \in (0,1)$.

$M(n,K)$ **is the least constant:**
$f \in QC_K(B^n), f(0) = 0, \ \forall (x, y \in B^n)$:
$|f(x) - f(y)| \le M(n,K)|x-y|^\alpha$.

Remark

1. $M(n,K) \to 1$, **when** $K \to 1$. **Hence Mori's constant 16 is not sharp.**
2. **S.-L. Qiu -97:** $M(2,K) \le 46^{1-1/K}$.
3. **Conjecture:** $M(2,K) \le 16^{1-1/K}$.

Theorem (R.Fehlmann-V -88)

$M(n,K) \le b(n,K) \to 1,$ **when** $K \to 1$**.**

Proof: Qc Schwarz lemma $+ \cdots$**.**

Theorem (Qc Schwarz lemma) [AVV, 1986]

$f \in QC_K(B^n), f(0) = 0 \Rightarrow$

$|f(x)| \le \phi_{K,n}(|x|) \le \lambda_n^{1-\alpha}|x|^\alpha$**.**

History, 1952-1986

Hersch-Pfluger $(\lambda_2 = 4)$**, Shabat, Gehring, Reshetnyak, Martio-Rickman-Väisälä, Anderson-Vamanamurthy-V .**

Hyperbolic metric ρ of B^n

$$\sinh^2 \frac{\rho(x,y)}{2} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}, \quad x, y \in B^n.$$

ρ **is Möbius invariant under selfmaps of** B^n.

Theorem (Bhayo-V 2009)

$f \in QC_K(B^2), x, y \in B^2 \Rightarrow$
$\rho(fx, fy) \le a(K) \max\{\rho(x,y)^{1/K}, \rho(x,y)\}$,
$a(K) < 1.36(K-1) + K$
($\to 1$, **when** $K \to 1$).

For the Schwarz lemma see also Epstein-Marden-Markovic 2004. For Lipschitz continuity of qc maps see Gutlyanskii-Gol'berg 2009.

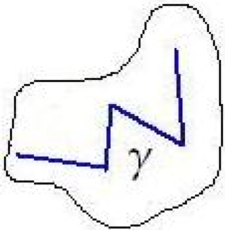
General metric space setup

$f : (G_1, d_1) \longrightarrow (G_2, d_2)$ **K-qc,** $G_1, G_2 \in \mathbb{R}^n$**.**

1. **Which choices** d_1, d_2 **are interesting?**
2. **Particular domains** G_1, G_2**.**

Quasihyperbolic metric

$G \subset \mathbb{R}^n$ **domain.** $d(x) = d(x, \partial G), x \in G$**.**
$\ell_k(\gamma) = \int_\gamma \frac{|\mathrm{d}x|}{d(x)}$**.** $k_G(x,y) = \inf_\gamma \ell_k(\gamma)$
(Gehring-Palka 1976). G-Osgood 1979: $\inf = \min$ **(i.e.**$\exists$ **geodesic).**
$k_{B^n}(x,y) \le \rho_{B^n}(x,y) \le 2k_{B^n}(x,y)$.



Theorem (GO -79)

$\exists c(n,K) : f : G \to G'$ is $K-$qc, $x, y \in G \Rightarrow k_{G'}(fx, fy) \le c \max\{k_G(x,y)^\alpha, k_G(x,y)\}$.

Remark

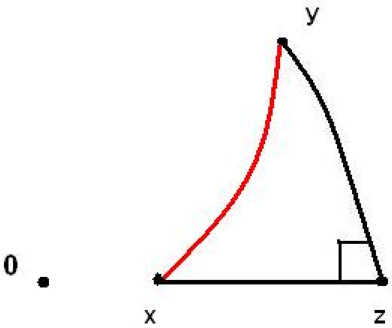
1. k_{B^n} **is not Möbius invariant.**
2. c **independent of** $G, c(n,1) > 1$.
3. **AVV 1988: Can choose** c **independent of** n**. For** $G = \mathbb{R}^n \setminus \{0\} : c \to \infty, \; K \to \infty$.

Distance formula (Martin-Osgood 1985)

$x, y \in G = \mathbb{R}^n \setminus \{0\}, \phi = \measuredangle([0,x],[0,y])$.

$$k_G(x,y)^2 = \phi^2 + \log^2 \frac{|x|}{|y|}, \ (MO).$$

Geodesics are logarithmic spirals. **Invariance under** $h(x) = x/|x|^2$: $k_G(hx, hy) = k_G(x,y)$. **Interpretation of (MO) as** k_G-**Pythagorean theorem** $(n=2)$**:** $k_G(x,y)^2 = k_G(y,z)^2 + k_G(z,x)^2, \ z = \frac{|y|}{|x|}x$.



$$z = x|y|/|x|$$

Klén 2009, $H^n = \{(x_1, ..., x_n) : x_n > 0\}$

Does the k_G Pythagorean thm hold as an inequality for some domain (e.g. $G = H^2$)?

Theorem (Klén-Manojlović-V 2010)

$\forall n \geq 2, K \in (1,2),\ \exists \omega(n,K) \to 1,$
when $K \to 1:$ **for**
$\forall x, y \in G = \mathbb{R}^n \setminus \{0\},\ f \in QC_K(G),$
$k_{fG}(fx, fy) \leq \omega \max\{k_G(x,y)^\alpha, k_G(x,y)\}$.

Note here the asymptotic sharpness $\omega(n,K) \to 1$, **when** $K \to 1$ **is essential.**

Ideas

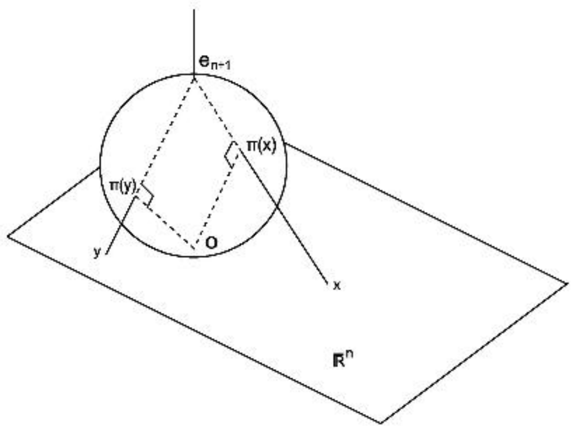
1. **Bernoulli type inequality.**
2. **Similar result for $j-$metric** $j_G(x,y) = \log(1 + \frac{|x-y|}{\min\{d(x),d(y)\}})$**.**
3. $\eta-$**quasisymmetry property of K-qc maps.**

Bernoulli type inequality

Let $0 < a \le 1 \le b,\ \phi(t) = \max\{t^a, t^b\}$.
Then for
$t > 0,\ \log(1+\phi(t)) \le \phi(\log(1+t))$.

Defined by stereographic projection

$\pi : q(x,y) = |\pi x - \pi y|, \ (\leq 1).$



Theorem (Beardon)

If $f : \overline{\mathbb{R}}^{n+1} \to \overline{\mathbb{R}}^{n+1}$ **is Möbius with** $fH^{n+1} = H^{n+1} = \{z \in \overline{\mathbb{R}}^{n+1} : z_{n+1} > 0\}$ **then with** $d = \rho_{H^{n+1}}(e_{n+1}, f(e_{n+1})),\ q(fx, fy) \le e^d q(x, y)\ \forall x, y$**.**

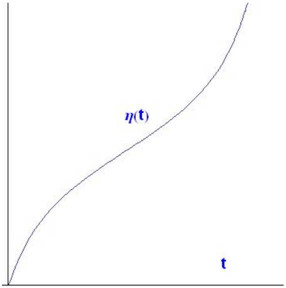
Theorem (Hästö 2004)

Let $f \in QC_K(\overline{\mathbb{R}}^{n+1})$, $f(0) = 0$, $fH^{n+1} = H^{n+1}$, $f(e_{n+1}) = e_{n+1}$, $f(\infty) = \infty$. **Then** $q(fx, fy) \le m(n, K)q(x, y)^{\alpha}$ $\forall x, y$. **Here** $m(n, K) \to 1$, **when** $K \to 1$.

Proof: nontrivial modification of the proof of Mori type theorems.

Definition

A homeo $f : \mathbb{R}^n \to \mathbb{R}^n$ is $\eta-$quasisymmetric if
$\forall x, y, z \in \mathbb{R}^n, x \neq z : \frac{|fx-fy|}{|fx-fz|} \leq \eta(\frac{|x-y|}{|x-z|})$.
Here a typical η is $\eta(t) = c \max\{t^{\alpha}, t^{1/\alpha}\}$



Theorem (V, 1990)

$f \in QC_K(\mathbb{R}^n) \Rightarrow f$ **is** $\eta-$**quasisymmetric with** $(\alpha = K^{1/(1-n)})$

$\eta(t) = c \max\{t^\alpha, t^{1/\alpha}\}, \;\; c \approx e^{100\sqrt{K-1}}$.

Lehto-Virtanen-Väisälä 1959: $t = 1$.

Open problem

Find counterparts for harmonic qc maps.

Harmonic qc: Martio 1968, Partyka-Sakan ca 1995, Kalaj-Pavlović 2005, Mateljević-V 2010

- R. Klén, V. Manojlović, and M. Vuorinen: **Distortion of normalized quasiconformal mappings.- arXiv:0808.1219 [math.CV], 20 pp. PDF**

- B. A. Bhayo and M. Vuorinen: **On Mori's theorem for quasiconformal maps in the *n*-space.- Trans. Amer. Math. Soc. (to appear) arXiv:0906.2853v4 [math.CA], 16pp. PDF**