Geometry of Carnot–Carathéodory Spaces and Applications

M. Karmanova, S. Vodopyanov

The talk is dedicated to investigation of Carnot — Carathéodory spaces. The geometry of Carnot–Carathéodory spaces naturally arises in physics, non-holonomic mechanics, contact geometry, theory of subelliptic equations, tomography, neurobiology, robotechnics etc. Heisenberg and Carnot groups are particular cases of Carnot–Carathéodory spaces.

We give some new crucial tools for the geometry of Carnot–Carathéodory spaces to investigate [2, 7].

One of underlying theorems in the theory of Carnot–Carathéodory spaces is Gromov's Theorem on nilpotentization of the basis vector fields. In literature, it is proved for the basis vector fields smooth enough. We prove it under minimal assumption of smoothness: when the basis vector fields are C^1-smooth. This result is applied for obtaining local estimates of closeness of two different local Carnot groups' geometries. These estimates are new even for the case of C^∞-vector fields; we prove it for C^1-vector fields [1, 7].

As an application of previous results, we estimate how the geometry of the initial Carnot–Carathéodory space is approximated by the one of a local Carnot group. We also develop our ideas and prove theorems that constitute foundations of Carnot–Carathéodory spaces' theory:

Gromov-type Local Approximation Theorem;

Chow–Rashevskiy Theorem;

Ball–Box Theorem;

Mitchell Theorem on Hausdorff dimension of Carnot–Carathéodory space for $C^{1,\alpha}$-smooth vector fields, $\alpha > 0$ [1, 7]. Earlier these results were proved for sufficiently smooth vector fields.

All above-mentioned is applied to investigate differentiability of mappings of Carnot–Carathéodory spaces. We obtain sub-Riemannian analogs of Rademacher's and Stepanov's Theorems and applications; these results are new [4, 7].

We use the obtained results and develop some new ideas for proving sub-Riemannian analogs of the area and coarea formulas for large classes of mappings of Carnot–Carathéodory spaces. The sub-Riemannian area formula is proved for Lipschitz (with respect to sub-Riemannian metric) mappings of Carnot–Carathéodory spaces. The result is new even for mappings of Carnot groups: we prove that

$$\int\limits_A \sqrt{\det\bigl(\widehat{D}\varphi(x)^*\widehat{D}\varphi(x)\bigr)}\, d\mathcal{H}^{\nu_1}(x) = \int\limits_{\varphi(A)} d\mathcal{H}^{\nu_1}(y),$$

where $\varphi \in \mathrm{Lip}(\mathbb{M}_1, \mathbb{M}_2)$, $A \subset \mathbb{M}_1$. Here the sub-Riemannian Jacobian equals $\sqrt{\det\bigl(\widehat{D}\varphi(x)^*\widehat{D}\varphi(x)\bigr)}$, where $\widehat{D}\varphi(x)$ is the $\mathcal{P}$-differential of φ at x. Note that, the methods of proving are new even for the case of a mapping of Euclidean spaces [5, 6].

The sub-Riemannian coarea formula [3]

$$\int\limits_{\mathbb{M}_1} \mathcal{J}_{N_2}^{SR}(\varphi, x)\, d\mathcal{H}^{\nu_1}(x) = \int\limits_{\mathbb{M}_2} d\mathcal{H}^{\nu_2}(t) \int\limits_{\varphi^{-1}(t)} d\mathcal{H}^{\nu_1-\nu_2}(u)$$

is proved for some classes of contact C^1-mappings of Carnot–Carathéodory spaces. Here the sub-Riemannian coarea factor equals

$$\mathcal{J}_{N_2}^{SR}(\varphi, x) = \sqrt{\det\bigl(\widehat{D}\varphi(x)\widehat{D}\varphi(x)^*\bigr)} \cdot \frac{\omega_{N_1}}{\omega_{\nu_1}} \frac{\omega_{\nu_2}}{\omega_{N_2}} \frac{\omega_{\nu_1-\nu_2}}{\prod\limits_{k=1}^{M_1} \omega_{n_k-\tilde{n}_k}}.$$

The result is new even for mappings of Carnot groups.

References

[1] Karmanova M., Vodopyanov S. *Local geometry of Carnot manifolds under minimal smoothness* // Dokl. AN, 2007. V. 413, № 3. P. 305–311.

[2] Karmanova M., Vodopyanov S. *Sub-Riemannian geometry under minimal smoothness of vector fields* // Dokl. AN, 2008, accepted.

[3] Karmanova M., Vodopyanov S. *The coarea formula for smooth contact mappings of Carnot manifolds* // Doklady Mathematics, 2007. V. 76, № 3. P. 908–912.

[4] Vodopyanov S. *Geometry of Carnot–Carathéodory Spaces and Differentiability of Mappings* // The Interaction of Analysis and Geometry. Contemporary Mathematics, **424** (2007), P. 247–302.

[5] Karmanova M., Vodopyanov S. *The area formula for contact C^1-mappings of Carnot–Carathéodory spaces* // Dokl. AN, 2008, submitted.

[6] Karmanova M. *The area formula for Lipschitz mappings of Carnot–Carathéodory spaces* // Dokl. AN, 2008, submitted.

[7] Karmanova M., Vodopyanov S. *Geometry of Carnot-Carathéodory spaces, differentiability and coarea formula* // In: Analysis and Mathematical Physics, Birkhäuser 2008, accepted.